

# CONVERGENCE OF LOGARITHMIC MEANS OF MULTIPLE WALSH-FOURIER SERIES

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**ABSTRACT.** The maximal Orlicz spaces such that the mixed logarithmic means of multiple Walsh-Fourier series for the functions from these spaces converge in measure and in norm are found.

Let  $\mathbb{I}^d := [0, 1]^d$  denote a cube in the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ . The elements of  $\mathbb{R}^d$  are denoted by  $\mathbf{x} := (x_1, \dots, x_d)$ .

Let  $D := \{1, 2, \dots, d\}$ ,  $B := \{l_1, l_2, \dots, l_r\}$ ,  $1 \leq r \leq d$ ,  $B \subset D$ ,  $l_k < l_{k+1}$ ,  $k = 1, 2, \dots, r-1$ ,  $B' := D \setminus B$ . For any  $\mathbf{x} = (x_1, \dots, x_d)$  and any  $B \subset D$ , denote  $\mathbf{x}_B := (x_{l_1}, x_{l_2}, \dots, x_{l_r}) \in \mathbb{R}^r$ . We assume that  $|B|$  is the number of elements of  $B$ . If  $B \neq \emptyset$ , then for any natural numbers  $n$  we suppose that  $n(B) := (n, n, \dots, n) \in \mathbb{R}^{|B|}$ . The notation  $a \lesssim b$  in the paper stands for  $a \leq cb$ , where  $c$  is an absolute constant. For any  $\mathbf{x} = (x_1, \dots, x_d)$  and  $\mathbf{y} = (y_1, \dots, y_d)$  the vector  $(x_1 \dot{+} y_1, \dots, x_d \dot{+} y_d)$  of the space  $\mathbb{R}^d$  is denoted by  $\mathbf{x} \dot{+} \mathbf{y}$ , where  $\dot{+}$  denotes the dyadic addition [10, 15].

Below we identify the symbols

$$\sum_{i_B=0_B}^{n_B} \quad \text{and} \quad \sum_{i_{l_1}=0}^{n_{l_1}} \cdots \sum_{i_{l_r}=0}^{n_{l_r}}, d\mathbf{t}_B \quad \text{and} \quad dt_{l_1} \cdots dt_{l_r}.$$

We denote by  $L_0(\mathbb{I}^d)$  the Lebesgue space of functions that are measurable and finite almost everywhere on  $\mathbb{I}^d$ .  $\text{mes}(A)$  is the Lebesgue measure of the set  $A \subset \mathbb{I}^d$ .

We denote by  $L_p(\mathbb{I}^d)$  the class of all measurable functions  $f$  that are 1-periodic with respect to all variable and satisfy

$$\|f\|_p := \left( \int_{\mathbb{I}^d} |f|^p \right)^{1/p} < \infty.$$

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The *weak* -  $L_1$  ( $\mathbb{I}^d$ ) space consist of all measurable, 1-periodic relative to each variables functions  $f$  for which

$$\|f\|_{weak-L_1(\mathbb{I}^d)} := \sup_{\lambda} \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : |f(\mathbf{x})| > \lambda \right\} < \infty.$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1 \quad \text{and} \quad x \in \mathbb{I}^1.$$

Let  $w_0, w_1, \dots$  represent the Walsh functions, i.e.  $w_0(x) = 1$ , and if  $n = 2^{n_1} + \dots + 2^{n_r}$  is a positive integer with  $n_1 > n_2 > \dots > n_r \geq 0$  then

$$w_n(x) = r_{n_1}(x) \cdots r_{n_r}(x).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$(1) \quad D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 1/2^n), \\ 0, & \text{if } x \in [1/2^n, 1). \end{cases}$$

The rectangular partial sums of d-dimensional Walsh-Fourier series are defined as follows:

$$S_{N_D}(f; \mathbf{x}) = \sum_{j_D=0_D}^{N_D-1_D} \widehat{f}(j_1, \dots, j_d) \prod_{i=1}^d w_{j_i}(x_i),$$

where the number

$$\widehat{f}(j_1, \dots, j_d) = \int_{\mathbb{I}^d} f(\mathbf{x}) \prod_{i=1}^d w_{j_i}(x_i) d\mathbf{x},$$

is said to be the  $(j_1, \dots, j_d)$ th Walsh-Fourier coefficient of the function  $f$ .

In the literature, it is known the notion of the Riesz's logarithmic means of a Fourier series. The  $n$ -th Riesz logarithmic mean of the Fourier series of the integrable function  $f$  is defined by

$$\frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{k}, \quad l_n := \sum_{k=1}^{n-1} \frac{1}{k},$$

where  $S_k(f)$  is the  $k$ th partial sum of its Fourier series. This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta [16, 18]. This mean is discussed first by Weisz [19] and then also by Simon and Gát [14, 2] with respect to the Walsh, Vilenkin system.

Let  $\{q_k : k \geq 0\}$  be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of  $f$  are defined by

$$\frac{1}{\sum_{k=1}^{n-1} q_k} \sum_{k=1}^{n-1} q_k S_{n-k}(f).$$

If  $q_k = \frac{1}{k}$ , then we get the (Nörlund) logarithmic means:

$$(2) \quad L_n(f; x) := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_{n-k}(f)}{k}.$$

Although, it is a kind of “reverse” Riesz’s logarithmic means. In [7] it is proved some convergence and divergence properties of the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions, and in the Lebesgue space  $L$ .

The Nörlund logarithmic and Reisz logarithmic means of multiple Fourier series are defined by

$$L_{n_D}(f; \mathbf{x}) := \frac{1}{\prod_{i \in D} l_i} \sum_{i_D=1_D}^{n_D-1_D} \frac{S_{n_D-i_D}(f; \mathbf{x})}{\prod_{j \in D} i_j},$$

$$R_{n_D}(f; \mathbf{x}) := \frac{1}{\prod_{i \in D} l_i} \sum_{i_D=0_D}^{n_D-1_D} \frac{S_{i_D}(f; \mathbf{x})}{\prod_{j \in D} i_j},$$

where  $n_D - i_D = (n_{l_1} - i_{l_1}, \dots, n_{l_r} - i_{l_r})$ . It is evident that

$$L_{n_D}(f; \mathbf{x}) = \int_{\mathbb{I}^d} f(\mathbf{t}) F_{n_D}(\mathbf{x} + \mathbf{t}) d\mathbf{t}$$

and

$$R_{n_D}(f; \mathbf{x}) = \int_{\mathbb{I}^d} f(\mathbf{t}) G_{n_D}(\mathbf{x} + \mathbf{t}) d\mathbf{t},$$

where

$$F_{n_D}(\mathbf{x}) := \prod_{j \in D} F_{n_j}(x_j), G_{n_D}(\mathbf{x}) := \prod_{j \in D} G_{n_j}(x_j),$$

$$F_n(u) := \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{D_{n-i}(u)}{i}, G_n(u) := \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{D_i(u)}{i}.$$

Let  $B \subset D$ . Then the mixed logarithmic means of multiple Walsh-Fourier series are defined by

$$(L_{n_B} \circ R_{n_{B'}})(f; \mathbf{x}) := \frac{1}{\prod_{i \in D} l_i} \sum_{i_D=1_D}^{n_D-1_D} \frac{S_{n_B-i_B, i_{B'}}(f; \mathbf{x})}{\prod_{j \in D} i_j}.$$

It is easy to show that

$$(L_{n_B} \circ R_{n_{B'}})(f; \mathbf{x}) = \int_{\mathbb{I}^d} f(\mathbf{t}) F_{n_B}(\mathbf{x}_B + \mathbf{t}_B) G_{n_{B'}}(\mathbf{x}_{B'} + \mathbf{t}_{B'}) d\mathbf{t}.$$

Let  $L_Q = L_Q(\mathbb{I}^d)$  be the Orlicz space [12] generated by Young function  $Q$ , i.e.  $Q$  is convex continuous even function such that  $Q(0) = 0$  and

$$\lim_{u \rightarrow +\infty} \frac{Q(u)}{u} = +\infty, \quad \lim_{u \rightarrow 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{I}^d)} = \inf\{k > 0 : \int_{\mathbb{I}^d} Q(|f|/k) \leq 1\}.$$

In particular, if  $Q(u) = u \log^\beta(1+u)$ ,  $u, \beta > 0$ , then the corresponding space will be denoted by  $L \log^\beta L(\mathbb{I}^d)$ .

In the trigonometric case the rectangular partial sums of double Fourier series  $S_{n,m}(f; x, y)$  of the function  $f \in L_p(\mathbb{T}^2)$ ,  $1 < p < \infty$ ,  $\mathbb{T} := [-\pi, \pi]$  converge in  $L_p$  norm to the function  $f$ , as  $n \rightarrow \infty$  [20]. In the case  $L_1(\mathbb{T}^2)$  this result does not hold. But for  $f \in L_1(\mathbb{T})$ , the operator  $S_n(f; x)$  are of weak type (1,1) [21]. This estimate implies convergence of  $S_n(f; x)$  in measure on  $\mathbb{T}$  to the function  $f \in L_1(\mathbb{T})$ . However, for double Fourier series this result does not hold [9, 13, 17]. Moreover, it is proved that quadratical partial sums  $S_{n,n}(f; x, y)$  of double Fourier series do not converge in two-dimensional measure on  $\mathbb{T}^2$  even for functions from Orlicz spaces wider than Orlicz space  $L \log L(\mathbb{T}^2)$ . On the other hand, it is well-known that if the function  $f \in L \log L(\mathbb{T}^2)$ , then rectangular partial sums  $S_{n,m}(f; x, y)$  converge in measure on  $\mathbb{T}^2$ .

Classical regular summation methods often improve the convergence of Fourier series. For instance, the Fejér means of the double Fourier series of the function  $f \in L_1(\mathbb{T}^2)$  converge in  $L_1(\mathbb{T}^2)$  norm to the function  $f$  [20]. These means present the particular case of the Nörlund means.

It is well known that the method of Nörlund logarithmic means of double Fourier series, is weaker than the Cesàro method of any positive order. In [3] it is proved, that these means of double Walsh-Fourier series in general do not converge in two-dimensional measure on  $\mathbb{I}^d$  even for functions from Orlicz spaces wider than Orlicz space  $L \log^{d-1} L(\mathbb{I}^d)$ . Thus, not all classic regular summation methods can improve the convergence in measure of double Fourier series.

The results for summability of logarithmic means of Walsh-Fourier series can be found in [4, 8, 6, 7, 16, 18].

In this paper we consider the mixed logarithmic means  $(L_{n_B} \circ R_{n_{B'}})(f)$  of rectangular partial sums multiple Walsh-Fourier series and prove that these means are acting from space  $L \log^{|B|} L(\mathbb{I}^d)$  into space  $L_1(\mathbb{I}^d)$  and from space  $L \log^{|B|-1} L(\mathbb{I}^d)$  into space *weak*  $-L_1(\mathbb{I}^d)$ . These facts implies

the convergence of mixed logarithmic means of rectangular partial sums of multiple Fourier series converge in norm and in measure (see Theorem 5). We also prove sharpness of these results. In particular, the following are true.

**Theorem 1.** *Let  $B \subset D$  and  $f \in L \log^{|B|} L(\mathbb{I}^d)$ . Then*

$$\|(L_{n_B} \circ R_{n_{B'}})(f)\|_{L_1(\mathbb{I}^d)} \lesssim 1 + \left\| |f| \log^{|B|} |f| \right\|_{L_1(\mathbb{I}^d)}.$$

**Theorem 2.** *Let  $B \subset D$  and  $f \in L \log^{|B|} L(\mathbb{I}^d)$ . Then*

$$\|(L_{n_B} \circ R_{n_{B'}})(f) - f\|_{L_1(\mathbb{I}^d)} \rightarrow 0 \quad \text{as } n_i \rightarrow \infty, i \in D;$$

**Theorem 3.** *Let  $L_Q(\mathbb{I}^d)$  be an Orlicz space, such that*

$$L_Q(\mathbb{I}^d) \not\subseteq L \log^{|B|} L(\mathbb{I}^d).$$

*Then*

*a)*

$$\sup_n \|L_{n(B)} \circ R_{n(B')}\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} = \infty;$$

*b) there exists a function  $f \in L_Q(\mathbb{I}^d)$  such that  $(L_{n(B)} \circ R_{n(B')})(f)$  does not converge to  $f$  in  $L_1(\mathbb{I}^d)$ -norm.*

Thus, the space  $L \log^{|B|} L(\mathbb{I}^d)$  is maximal Orlicz space such that for each function  $f$  from this space the means  $(L_{n(B)} \circ R_{n(B')})(f)$  converge to  $f$  in  $L_1(\mathbb{I}^d)$ -norm.

**Theorem 4.** *Let  $B \subset D$  and  $f \in L \log^{|B|-1} L(\mathbb{I}^d)$ . Then*

$$\|(L_{n_B} \circ R_{n_{B'}})(f)\|_{weak-L_1(\mathbb{I}^d)} \lesssim 1 + \left\| |f| \log^{|B|-1} |f| \right\|_{L_1(\mathbb{I}^d)}.$$

**Theorem 5.** *Let  $B \subset D$  and  $f \in L \log^{|B|-1} L(\mathbb{I}^d)$ . Then*

$$(L_{n_B} \circ R_{n_{B'}})(f) \rightarrow f \text{ in measure on } \mathbb{I}^d, \text{ as } n_i \rightarrow \infty, i \in D.$$

**Theorem 6.** *Let  $B \subset D, |B| > 1$  and  $L_Q(\mathbb{I}^d)$  be an Orlicz space, such that*

$$L_Q(\mathbb{I}^d) \not\subseteq L \log^{|B|-1} L(\mathbb{I}^d).$$

*Then the set of the functions from the Orlicz space  $L_Q(\mathbb{I}^d)$  with logarithmic means  $(L_{n_B} \circ R_{n_{B'}})(f)$  of rectangular partial sums of multiple Fourier series convergent in measure on  $\mathbb{I}^d$  is of first Baire category in  $L_Q(\mathbb{I}^d)$ .*

**Corollary 1.** *Let  $B \subset D$ ,  $|B| > 1$  and  $\varphi : [0, \infty[ \rightarrow [0, \infty[$  be a nondecreasing function satisfying for  $x \rightarrow +\infty$  the condition*

$$\varphi(x) = o(x \log^{|B|-1} x).$$

*Then there exists the function  $f \in L_1(\mathbb{I}^d)$  such that*

a)

$$\int_{\mathbb{I}^d} \varphi(|f|) < \infty;$$

b) *logarithmic means  $(L_{n_B} \circ R_{n_{B'}})(f)$  of rectangular partial sums of multiple Fourier series of  $f$  diverges in measure on  $\mathbb{I}^d$ .*

## 1. AUXILIARY RESULTS

We apply the reasoning of [1] formulated as the following proposition in particular case.

**Theorem 7.** *Let  $H : L_1(\mathbb{I}^d) \rightarrow L_0(\mathbb{I}^d)$  be a linear continuous operator, which commutes with family of translations  $\mathcal{E}$ , i. e.  $\forall E \in \mathcal{E} \quad \forall f \in L_1(\mathbb{I}^d) \quad HEf = EHf$ . Let  $\|f\|_{L_1(\mathbb{I}^d)} = 1$  and  $\lambda > 1$ . Then for any  $1 \leq r \in \mathbb{N}$  under condition  $\text{mes}\{\mathbf{x} \in \mathbb{I}^d : |Hf| > \lambda\} \geq \frac{1}{r}$  there exist  $E_1, \dots, E_r \in \mathcal{E}$  and  $\varepsilon_i = \pm 1$ ,  $i = 1, \dots, r$  such that*

$$\text{mes}\{\mathbf{x} \in \mathbb{I}^d : |H(\sum_{i=1}^r \varepsilon_i) f(E_i \mathbf{x})| > \lambda\} \geq \frac{1}{8}.$$

**Lemma 1.** *Let  $\{H_m\}_{m=1}^\infty$  be a sequence of linear continuous operators, acting from Orlicz space  $L_Q(\mathbb{I}^d)$  in to the space  $L_0(\mathbb{I}^d)$ . Suppose that there exists a sequence of functions  $\{\xi_k\}_{k=1}^\infty$  from unit ball  $S_Q(0, 1)$  of space  $L_Q(\mathbb{I}^d)$ , sequences of integers  $\{m_k\}_{k=1}^\infty$  and  $\{\nu_k\}_{k=1}^\infty$  increasing to infinity such that*

$$\varepsilon_0 = \inf_k \text{mes}\{\mathbf{x} \in \mathbb{I}^d : |H_{m_k} \xi_k(x, y)| > \nu_k\} > 0.$$

*Then  $K$  - the set of functions  $f$  from space  $L_Q(\mathbb{I}^d)$ , for which the sequence  $\{H_m f\}$  converges in measure to an a. e. finite function is of first Baire category in space  $L_Q(\mathbb{I}^d)$ .*

The proof of Lemma 1 can be found in [3].

**Lemma 2.** *Let  $L_\Phi(\mathbb{I}^d)$  be an Orlicz space and let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be measurable function with condition  $\varphi(x) = o(\Phi(x))$  as  $x \rightarrow \infty$ . Then there exists Orlicz space  $L_\omega(\mathbb{I}^d)$ , such that  $\omega(x) = o(\Phi(x))$  as  $x \rightarrow \infty$ , and  $\omega(x) \geq \varphi(x)$  for  $x \geq c \geq 0$ .*

The proof of Lemma 2 can be found in [4].

Set  $m_* := \lfloor m/2 \rfloor$  (the lower integer part of  $m/2$ )

$$\tilde{m} := \left\lfloor \frac{l_{p_{m_*}-1}}{16} - 2^{15} \right\rfloor, J_m := \left[ \frac{1}{2^{m+1}}, \frac{1}{2^{m+1}} + \frac{1}{2^{m+\tilde{m}}} \right),$$

$$J := \bigcup_{m=m_0}^{\infty} J_m, m_0 := \inf \left\{ m : \left\lfloor \frac{l_{p_{m_*}-1}}{16} - 2^{15} \right\rfloor > 1 \right\},$$

$$\Omega_n := \bigcup_{m=n}^{2n} \left[ \frac{1}{2^{m+1}} + \frac{1}{2^{m+\tilde{m}}}, \frac{1}{2^m} \right),$$

where  $p_n := 2^{2n} + 2^{2n-2} + \dots + 2^0$ .

**Lemma 3.** *For  $x \in \Omega_n$  we have an estimation*

$$|F_{p_n}(x)| \gtrsim \frac{1}{x}.$$

The proof of Lemma 3 can be found in [3].

## 2. PROOF OF THE THEOREMS

*Proof of Theorem 1.* We apply the following particular case of the Marcinkiewicz interpolation theorem [11]. Let  $T : L_1(\mathbb{I}^1) \rightarrow L_0(\mathbb{I}^1)$  be a quasilinear operator of weak type  $(1, 1)$  and of type  $(\alpha, \alpha)$  for some  $1 < \alpha < \infty$  at the same time, i. e.

$$(3) \quad \begin{aligned} & \text{mes} \{x \in \mathbb{I}^1 : |T(f, x)| > y\} \\ & \lesssim \frac{1}{y} \int_{\mathbb{I}^1} |f(x)| dx; \quad \forall f \in L_1(\mathbb{I}^1) \quad \forall y > 0; \end{aligned}$$

and

$$(4) \quad \|Tf\|_{L_\alpha(\mathbb{I}^1)} \lesssim \|f\|_{L_\alpha(\mathbb{I}^1)}, \quad \forall f \in L_\alpha(\mathbb{I}^1).$$

Then

$$(5) \quad \begin{aligned} & \int_{\mathbb{I}^1} |T(f, x)| \ln^\beta |T(f, x)| dx \\ & \lesssim \int_{\mathbb{I}^1} |f(x)| \ln^{\beta+1} |f(x)| dx + 1, \quad \forall \beta \geq 0. \end{aligned}$$

In [5] it is proved that for any  $f \in L_1(\mathbb{I}^1)$  the operator  $f * F_n$  has weak type  $(1, 1)$ , i. e.

$$(6) \quad \|f * F_n\|_{weak\_L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}.$$

On the other hand, it is easy to prove that the operator  $f * G_n$  has type  $(1, 1)$ , i.e.

$$(7) \quad \|f * G_n\|_{L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}.$$

From (3)-(7) we have ( $B' := \{s_1, s_2, \dots, s_{r'}\}$ )

$$\begin{aligned}
(8) \quad & \left\| (L_{n_B} \circ R_{n_{B'}})(f) \right\|_{L_1(\mathbb{I}^d)} \\
&= \left\| \left( R_{n_{s_1}} \circ \dots \circ R_{n_{s_{r'}}} \circ L_{n_{l_1}} \circ \dots \circ L_{n_{l_r}} \right)(f) \right\|_{L_1(\mathbb{I}^d)} \\
&\lesssim \dots \lesssim \left\| \left( L_{n_{l_1}} \circ \dots \circ L_{n_{l_r}} \right)(f) \right\|_{L_1(\mathbb{I}^d)} \\
&\lesssim 1 + \left\| \left| L_{n_{l_2}} \circ \dots \circ L_{n_{l_r}}(f) \right| \log \left| L_{n_2} \circ \dots \circ L_{n_{l_r}}(f) \right| \right\|_{L_1(\mathbb{I}^d)} \\
&\lesssim \dots \lesssim 1 + \left\| \left| L_{n_{l_r}}(f) \right| \log^{r-1} \left| L_{n_{l_r}}(f) \right| \right\|_{L_1(\mathbb{I}^d)} \\
&\lesssim 1 + \left\| |f| \log^r |f| \right\|_{L_1(\mathbb{I}^d)}.
\end{aligned}$$

Theorem 1 is proved.  $\square$

By virtue of standart arguments (see [21]) we can see the validity of Theorem 2.

*Proof of Theorem 3.* Let

$$Q\left(2^{2n|B|}\right) \gtrsim 2^{2n|B|} \quad \text{for } n > n_0.$$

By virtue of estimate ([12], Ch. 2)

$$\|f\|_{L_Q(\mathbb{I}^d)} \leq 1 + \|Q(|f|)\|_{L_1(\mathbb{I}^d)}.$$

We can write

$$\begin{aligned}
(9) \quad & \left\| L_{p_n(B)} \circ R_{p_n(B')} \left( \bigotimes_{i \in B} \frac{D_{2^{2n+1}}}{2} \right) \right\|_{L_1(\mathbb{I}^d)} \\
&\leq \left\| L_{p_n(B)} \circ R_{p_n(B')} \right\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} \left\| \bigotimes_{i \in B} \frac{D_{2^{2n+1}}}{2} \right\|_{L_Q(\mathbb{I}^d)} \\
&\leq \left\| L_{p_n(B)} \circ R_{p_n(B')} \right\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} \\
&\quad \times \left( 1 + \left\| Q \left( \bigotimes_{i \in B} \frac{D_{2^{2n+1}}}{2} \right) \right\|_{L_1(\mathbb{I}^d)} \right) \\
&\lesssim \left\| L_{p_n(B)} \circ R_{p_n(B')} \right\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} \left( 1 + \frac{1}{2^{2n|B|}} Q\left(2^{2n|B|}\right) \right) \\
&\lesssim \left\| L_{p_n(B)} \circ R_{p_n(B')} \right\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} \frac{Q\left(2^{2n|B|}\right)}{2^{2n|B|}}.
\end{aligned}$$



On the other hand,

$$\begin{aligned}
(10) \quad & L_{p_n(B)} \circ R_{p_n(B')} \left( \bigotimes_{i \in B} \frac{D_{2^{2n+1}}}{2}; \mathbf{x} \right) \\
&= \frac{1}{2^{|B|}} \int \prod_{i \in B} D_{2^{2n+1}}(z_i) F_{p_n}(x_i + z_i) d\mathbf{z}_B \\
&\quad \times \int \prod_{j \in B'} G_{p_n}(x_j + z_j) d\mathbf{z}_{B'} \\
&= \frac{1}{2^{|B|}} \int \prod_{i \in B} D_{2^{2n+1}}(z_i) F_{p_n}(x_i + z_i) d\mathbf{z}_B \\
&= \frac{1}{2^{|B|}} \prod_{i \in B} \int_{\mathbb{I}} D_{2^{2n+1}}(z_i) F_{p_n}(x_i + z_i) dz_j \\
&= \frac{1}{2^{|B|}} \prod_{i \in B} S_{2^{2n+1}}(F_{p_n}; x_i) \\
&= \frac{1}{2^{|B|}} \prod_{i \in B} F_{p_n}(x_i).
\end{aligned}$$

Consequently, from Lemma 3 we get

$$\begin{aligned}
(11) \quad & \left\| L_{p_n(B)} \circ R_{p_n(B')} \left( \bigotimes_{i \in B} \frac{D_{2^{2n+1}}}{2}; \mathbf{x} \right) \right\|_{L_1(\mathbb{I}^d)} \\
&= \frac{1}{2^{|B|}} \prod_{i \in B} \|F_{p_n}(x_i)\|_{L_1(\mathbb{I}^1)} \gtrsim n^{|B|}.
\end{aligned}$$

Combining (9) and (11) we obtain

$$(12) \quad \|L_{p_n(B)} \circ R_{p_n(B')}\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} \gtrsim \frac{2^{2n|B|} n^{|B|}}{Q(2^{2n|B|})}.$$

The fact that

$$L_Q(\mathbb{I}^d) \not\subseteq L \log^{|B|} L(\mathbb{I}^d)$$

is equivalent to the condition

$$\limsup_{u \rightarrow \infty} \frac{u \log^{|B|} u}{Q(u)} = \infty.$$

Thus, there exists  $\{u_k : k \geq 1\}$  such that

$$\lim_{k \rightarrow \infty} \frac{u_k \log^{|B|} u_k}{Q(u_k)} = \infty, u_{k+1} > u_k, k = 1, 2, \dots,$$

and a monotonically increasing sequence of positive integers  $\{r_k : k \geq 1\}$  such that

$$2^{2|B|r_k} \leq u_k < 2^{2|B|(r_k+1)}.$$

Then we have

$$\frac{2^{2r_k|B|} r_k^{|B|}}{Q(2^{2r_k|B|})} \gtrsim \frac{u_k \log^{|B|} u_k}{Q(u_k)} \rightarrow \infty.$$

Thus, from (12) we conclude that

$$\sup_n \|L_{p_n(B)} \circ R_{p_n(B')}\|_{L_Q(\mathbb{I}^d) \rightarrow L_1(\mathbb{I}^d)} = \infty.$$

This completes the proof of Theorem 3 part a). Part b) follows immediately from part a).  $\square$

*Proof of Theorem 4.* Set

$$\Omega := \left\{ \mathbf{x} \in \mathbb{I}^d : |(L_{n_B} \circ R_{n_{B'}})(f, \mathbf{x})| > \lambda \right\}.$$

Then from (3)-(8) we have

$$\begin{aligned} & \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : |(L_{n_B} \circ R_{n_{B'}})(f, \mathbf{x})| > \lambda \right\} \\ &= \int_{\mathbb{I}^d} \mathbf{1}_\Omega(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{I}^{d-1}} \left( \int_{\mathbb{I}} \mathbf{1}_\Omega(\mathbf{x}) dx_{l_1} \right) d\mathbf{x}_{D \setminus \{l_1\}} \\ &\lesssim \frac{1}{\lambda} \left\| (L_{n_{B \setminus \{l_1\}}} \circ R_{B'}) (f) \right\|_{L_1(\mathbb{T}^d)} \\ &\lesssim 1 + \| |f| \log^{r-1} |f| \|_{L_1(\mathbb{T}^d)}. \end{aligned}$$

Theorem 4 is proved.  $\square$

By virtue of standart arguments (see [21]) we can see the validity of Theorem 5.

*Proof of Theorem 6.* By Lemma 1 the proof of Theorem 3 will be complete if we show that there exists sequences of integers  $\{n_k : k \geq 1\}$  and  $\{\nu_k : k \geq 1\}$  increasing to infinity, and a sequence of functions  $\{\xi_k : k \geq 1\}$  from the unit ball  $S_Q(0, 1)$  of Orlicz space  $L_Q(\mathbb{I}^d)$ , such that for all  $k$

$$(13) \quad \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : \left| L_{p_{n_k}(B)} \circ R_{p_{n_k}(B')} (\xi_k; \mathbf{x}) \right| > \nu_k \right\} \geq \frac{1}{8}.$$

First, we prove that

$$\begin{aligned} (14) \quad & \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : \left| L_{p_n(B)} \circ R_{p_n(B')} \left( \bigotimes_{i \in B} D_{2^{2n+1}}; \mathbf{x} \right) \right| \gtrsim 2^{n(2|B|-1)} \right\} \\ & \gtrsim \frac{n^{|B|-1}}{2^{n(2|B|-1)}}, \quad |B| > 1. \end{aligned}$$

From (10) and Lemma 3 we have

$$\begin{aligned}
& \left| L_{p_n(B)} \circ R_{p_n(B')} \left( \bigotimes_{i \in B} D_{2^{2n+1}}; \mathbf{x} \right) \right| \\
&= \prod_{i \in B} |F_{p_n}(x_i)| \\
&\gtrsim \prod_{j \in B} \frac{1}{x_j}, x_j \in \Omega_n, \quad j \in B.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : \left| L_{p_n(B)} \circ R_{p_n(B')} \left( \bigotimes_{i \in B} D_{2^{2n+1}}; \mathbf{x} \right) \right| \gtrsim 2^{n(2|B|-1)} \right\} \\
&\geq \text{mes} \left\{ \mathbf{x} \in \Omega_n^{|B|} \times \mathbb{I}^{|B'|} : \prod_{j \in B} \frac{1}{x_j} \gtrsim 2^{n(2|B|-1)} \right\} \\
&= \text{mes} \left\{ \mathbf{x}_B \in \Omega_n^{|B|} : x_{l_1} \lesssim \frac{1}{2^{n(2|B|-1)} \prod_{j \in B \setminus \{l_1\}} x_j} \right\} \\
&= \sum_{m_B = n(B)}^{2n(B)} \text{mes} \left\{ x_j \in \left[ \frac{1}{2^{m_j+1}} + \frac{1}{2^{m_j+\tilde{m}_j}}, \frac{1}{2^{m_j}} \right), j \in B : \right. \\
&\quad \left. x_{l_1} \lesssim \frac{1}{2^{n(2|B|-1)} \prod_{j \in B \setminus \{l_1\}} x_j} \right\} \\
&\gtrsim \sum_{m_{B \setminus \{l_1\}} = n(B \setminus \{l_1\})}^{2n(B \setminus \{l_1\})} \sum_{m_{l_1} = n(2|B|-1) - (m_{l_2} + \dots + m_{l_r})}^{2n} \prod_{j \in B} \frac{1}{2^{m_j}} \\
&\gtrsim \prod_{j \in B \setminus \{l_1\}} \sum_{m_j = n}^{2n} \frac{1}{2^{m_j}} \sum_{m_{l_1} = n(2|B|-1) - (m_{l_2} + \dots + m_{l_r})}^{2n} \frac{1}{2^{m_{l_1}}} \\
&\gtrsim \frac{1}{2^{n(2|B|-1)}} \prod_{j \in B \setminus \{l_1\}} \sum_{m_j = n}^{2n} \frac{2^{m_j}}{2^{m_j}} \\
&\gtrsim \frac{n^{|B|-1}}{2^{n(2|B|-1)}}, \quad |B| > 1.
\end{aligned}$$

Here we also used that for  $n(2|B|-1) - (m_{l_2} + \dots + m_{l_r}) \leq m_{l_1} \leq 2n$  we have

$$x_{l_1} \leq \frac{1}{2^{m_{l_1}}} \leq \frac{2^{m_{l_2} + \dots + m_{l_r}}}{2^{n(2|B|-1)}} \leq \frac{1}{2^{n(2|B|-1)} x_{l_2} \dots x_{l_r}}.$$

Hence (14) is proved.

From the condition of the theorem we write

$$\liminf_{u \rightarrow \infty} \frac{Q(u)}{u \log^{|B|-1} u} = 0.$$

Consequently, there exists a sequence of integers  $\{n_k : k \geq 1\}$  increasing to infinity, such that

$$(15) \quad \lim_{k \rightarrow \infty} \frac{Q(2^{2n_k|B|})}{2^{2n_k|B|} n_k^{|B|-1}} = 0, \quad \frac{Q(2^{2n_k|B|})}{2^{|B|(2n_k+1)}} \geq 1, \quad \forall k.$$

From (14) we have

$$\begin{aligned} & \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : \left| L_{p_{n_k}(B)} \circ R_{p_{n_k}(B')} \left( \bigotimes_{i \in B} D_{2^{2n_k+1}}; \mathbf{x} \right) \right| \gtrsim 2^{n_k(2|B|-1)} \right\} \\ & \gtrsim \frac{n_k^{|B|-1}}{2^{n_k(2|B|-1)}}. \end{aligned}$$

Then by the virtue of Theorem 7 there exists  $E_1, \dots, E_r \in \mathcal{E}$  and  $\varepsilon_1, \dots, \varepsilon_r = \pm 1$  such that

$$(16) \quad \begin{aligned} & \text{mes} \left\{ \mathbf{x} \in \mathbb{I}^d : \left| \sum_{i=1}^{r_k} \varepsilon_i L_{p_{n_k}(B)} \circ R_{p_{n_k}(B')} \left( \bigotimes_{j \in B} D_{2^{2n_k+1}}; E_i \mathbf{x} \right) \right| \right. \\ & \left. > 2^{n_k(2|B|-1)} \right\} > \frac{1}{8}, \end{aligned}$$

where

$$r_k \sim \frac{2^{n_k(2|B|-1)}}{n_k^{|B|-1}}.$$

Denote

$$\nu_k = \frac{2^{n_k(4|B|-1)-1}}{r_k Q(2^{2n_k|B|})}$$

and

$$\xi_k(\mathbf{x}) = \frac{2^{2|B|n_k-1}}{Q(2^{2n_k|B|})} M_k(\mathbf{x}),$$

where

$$\begin{aligned} M_k(\mathbf{x}) &= \frac{1}{r_k} \sum_{i=1}^{r_k} \varepsilon_i \prod_{j \in B} D_{2^{2n_k+1}} \left( E_i^{(j)} x_j \right), \\ E_i &:= \left( E_i^{(1)}, \dots, E_i^{(d)} \right) \end{aligned}$$

Thus, from (16) we obtain (13).

Finally, we prove that  $\xi_k \in S_Q(0, 1)$ . Since

$$\begin{aligned} \|M_k\|_\infty &\leq 2^{|B|(2n_k+1)}, \\ \|M_k\|_{L_1(\mathbb{I}^d)} &\leq 1, \end{aligned}$$

$$\|\xi_k\|_{L_Q(\mathbb{I}^d)} \leq \frac{1}{2} \left[ \int_{\mathbb{I}^d} Q(2|\xi_k|) + 1 \right],$$

and

$$\frac{Q(u)}{u} < \frac{Q(u')}{u'}, \quad (0 < u < u')$$

we can write

$$\begin{aligned} \|\xi_k\|_{L_Q(\mathbb{I}^d)} &\leq \frac{1}{2} \left[ 1 + \int_{\mathbb{I}^d} Q\left(\frac{2^{2|B|n_k} |M_k(x)|}{Q(2^{2|B|n_k})}\right) d\mathbf{x} \right] \\ &\leq \frac{1}{2} \left[ 1 + \int_{\mathbb{I}^d} \frac{Q\left(\frac{2^{2|B|n_k} 2^{B|(2n_k+1)}}{Q(2^{2|B|n_k})}\right)}{\frac{2^{2|B|n_k} 2^{B|(2n_k+1)}}{Q(2^{2|B|n_k})}} \frac{2^{2|B|n_k} |M_k(\mathbf{x})|}{Q(2^{2|B|n_k})} d\mathbf{x} \right] \\ &\leq \frac{1}{2} \left[ 1 + \int_{\mathbb{I}^d} \frac{Q(2^{2|B|n_k})}{2^{2|B|n_k}} \frac{2^{2|B|n_k} |M_k(\mathbf{x})|}{Q(2^{2|B|n_k})} d\mathbf{x} \right] \\ &\leq 1. \end{aligned}$$

Hence,  $\xi_k \in S_Q(0, 1)$ , and Theorem 6 is proved.  $\square$

The validity of Corollary 1 follows immediately from Theorem 6 and Lemma 2.

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